UNITED STATES
NAVAL POSTGRADUATE SCHOOL

ON THE
IMPEDANCE OF A RADIATING CIRCUIT

---BY---

JESSE GERALD CHANEY
PROFESSOR OF ELECTRONICS

A REPORT
TO THE NAVY DEPARTMENT
BUREAU OF SHIPS
UPON AN INVESTIGATION CONDUCTED UNDER
BUSHIPS PROJECT ORDER NO. 10731/52

MARCH, 1953

TECHNICAL REPORT NO. 9

UNITED STATES

NAVAL POSTGRADUATE SCHOOL

ON THE IMPEDANCE OF A RADIATING CIRCUIT

---BY---

JESSE GERALD CHANEY
PROFESSOR OF ELECTRONICS

REPORT NO. 6

ON THE IMPEDANCE OF A RADIATING CIRCUIT

*Jesse Gerald Chaney**
United States Naval Postgraduate School
Monterey, California

ABSTRACT

In applying the formula for the driving point impedance of a radiating electric circuit, as given by the generalized circuit method, it is found that additional terms must be included. These terms are required to account for the negative spatial rate of change at which the reactive power is being returned into the wire from the localized principal wave, retardation being neglected. In the case of a biconical antenna, it is demonstrated that these terms are cancelled by choosing a proper procedure of integration. However, these terms must be retained in the case of a radiating transmission line. The radiation impedance of a parallel wire line is given and is uniformly distributed along the line. The added attenuation due to radiation losses is considered in the determination of the driving point impedance of the line. The latter procedure is applicable to a terminated rhombic antenna provided due cognizance is taken of the non-uniformity of the line.

INTRODUCTION

Since the generalized electric circuit as previously presented by the writer[1] is derived from a complex Poynting vector method, it is basically equivalent to other Poynting vector and induced *emf* methods for determining the power radiated by the circuit. However, the formal integral equation for the driving point impedance of a wire antenna differs from the customary formulation in that the real part of the product of the current distribution function with its conjugate is taken within the integrand. Using only the real part of the conjugate current product insures that the reciprocity of the mutual impedances will follow when a complex current distribution is postulated *a priori*, and it also facilitates the

---

* Professor of Electronics

1. J. G. Chaney, 'A critical study of the circuit concept', J. Appl. Phys. 22, 12, 1429, (1951).

integrations for the radiation impedance of the circuit.

The expression *radiation impedance* is used by Schelkunoff and Friis[2] for the reaction of the wave produced by the current. However, the reaction in question is usually considered to be due to the effects of retardation.

Thus, it seems that the radiation impedance is due to the higher order fields supported by the circuit. If referred to the driving point current, the resistive component of the radiation impedance may be used in determining the efficiency of a circuit as a radiator of electromagnetic energy. Likewise, the reactive component of the radiation impedance produces an additional reactive term in the driving point impedance, this term being due to the effects of radiations from the circuit. In other words, the effects of retardation along the circuit produces additional reactance in the impedance of the circuit as well as apparent additional resistance.

The radiation resistance of a radiating circuit may or may not appear directly in the driving point impedance. In the asymtotic form for a postulated real current distribution, it appears directly. However, for a terminated transmission line on which travelling waves are postulated, it enters through the attenuation of the wave.

In deriving the formula for the driving point impedance of a generalized circuit[1], it was assumed that the reactive power per unit length in the internal fields of a cross section of the wire was produced solely by the non-retarded interior fields, the latter fields being produced by the current density through the cross section itself. However, in applying the theory to the problem of determining the driving point impedance of a radiating, terminated transmission line, it was discovered that a part of the reactive component of the driving point impedance was missing. It will be shown that this discrepancy is due to the failure to include the spatial rate of change at which the reactive power due to the localized non-retarded fields is re-entering the wire. In other words, using the arc length coordinate s along the wire, a correction term must be included in

2. S. A. Schelkunoff and H. T. Friis, 'Antennas Theory and Practice', Jno. Wiley and Sons, N. Y., 1952.

the reactive power, namely,

$$-j\ \tfrac{1}{2}\ |I_0|^2 \frac{\partial}{\partial s}\ [\,|f(s)|^2 X_p(s)\,]$$

in which

$I_0 f(s)$ = *current distribution function with* $f(0) = 1$
$X_p(s)$ = *the imaginary part of the ratio of the voltage to the current in the principal wave*

DRIVING POINT IMPEDANCE

The formula for the driving point impedance of a generalized circuit (*Figure 1*) having its generator at terminals (b,c) was previously given as[1]

$$Z_{in} = lZ_i f_m{}^2 + j(30/k)\int_e^d\int_e^d Re\,[f(P_1)^* f(P_2)]\,\tilde{\nabla}_1\,[e(r_{21})\,d\bar{r}_2]\cdot d\bar{r}_1 \qquad (1)$$

in which

Z_i = *internal impedance per unit length*
P_1 = *any point on the axis of the wire*
P_2 = *any point on the inner periphery of the wire*
l = *length of the circuit*
$k = \omega\sqrt{\mu_0 \epsilon_0} = 2\pi/\lambda$ = *wave number*
μ_0 = *permeability of free space*
ϵ_0 = *permitivity of free space*
$f_m{}^2$ = *mean square current distribution*
r_{21} = *distance from* P_1 *to* P_2
$e(r_{21}) = r_{21}^{-1}\exp(-jkr_{21})$ = *retardation function*
$\tilde{\nabla}_1 = \nabla_1 \nabla_1 \cdot + k^2$ = *field operator deltil with subscript indicating positions at which differentiations are to be performed*
Re = *real part to be taken*
$*$ = *complex conjugate to be taken*

The terms involving the internal impedance per unit length was determined by neglecting retardation and integrating for the complex power over a cross section of the wire, thus removing the singularities from the retarded volume integral from which the second term was derived. However, the approximate form of the second term avoids singularities by following the conventional method of choosing the paths of integration after replacing the current densities by currents, unless the wires are tapered such as in the case of a bi-conical antenna. Since the first term

excludes retardation and the second term includes retardation while excluding non-retardation, it follows that the second term gives the radiation impedance of the circuit. It also follows that the first term is due to the principal wave guided by the circuit and that the second term is due to the higher order waves guided by the circuit[2,3].

In determining the internal impedance per unit length, the following equivalent expression was obtained for the complex power per unit length[1],

$$W_{01} = \tfrac{1}{2}|I_0|^2|f(s)|^2 Z_i - \tfrac{1}{2}\int_{S'}\bar{E}_i\cdot\bar{\imath}^* dS' - j\omega(|U_{H1}|-|U_{E1}|) \qquad (2)$$

where

W_{01} = *complex power per unit length*

$\bar{E}_i$ = *internal field due to the current density through the cross section*

$\bar{\imath}$ = *current density through the cross section*

U_{H1} = *peak energy per unit length stored in the magnetic field within the wire*

U_{E1} = *peak energy stored in the electric field per unit length within the wire*

By further letting

$\bar{E}_1$ = *the total electric field per unit length*

$\bar{E}_{01}$ = *the applied field per unit length*

$W_{1\,av}$ = *time average power per unit length dissipated within the wire*

$Z_i = R_i + jX_i$

then since

$$W_{01} = \tfrac{1}{2}\int_{S'}\bar{E}_{01}\cdot\bar{\imath}^*\, dS' \qquad (3)$$

it follows that equation (2) may be written as

$$\tfrac{1}{2}\int_{S'}\bar{E}\cdot\bar{\imath}^*\, dS' = -\frac{\partial}{\partial s} W_{1\,av} = \tfrac{1}{2}|I_0 f(s)|^2 R_i \qquad (4)$$

Hence

$$\tfrac{1}{2}\int_S \bar{E}\cdot\bar{\imath}^*\, dS' = \tfrac{1}{2}|I_0 f(s)|^2 Z_i - j\omega(|U_{H1}-|U_{E1}|) \qquad (5)$$

---

3. S. A. Schelkunoff, 'Theory of antennas of arbitrary size and shape', Proc. I. R. E. 29, 9, 493, Sept., 1941

or

$$X_i = (2\omega/|I_0 f(s)|^2)(|U_{H1}|-|U_{E1}|) \quad (6)$$

From equations (3) and (6), the complex power per unit length becomes

$$W_{01} = \tfrac{1}{2}|I_0 f(s)|^2 R_i - \tfrac{1}{2}\int_{S'} \bar{E}_{i1}\cdot\bar{i}^* dS' \quad (7)$$

Thus, from equation (7) it appears that the second and third terms in equation (2) are equivalent, as previously assumed, only if the reactive power per unit length entering the wire from the localized external fields vanishes. Hence, it follows more generally that

$$-\tfrac{1}{2}\int_{S'} \bar{E}_{i1}\cdot\bar{i}^* dS' = j\tfrac{1}{2}|I_0 f(s)|^2 - j\omega\frac{\partial}{\partial s}\left[|U_{Hp}|-|U_{Ep}|\right] \quad (8)$$

where $|U_{Hp}|$ *and* $|U_{Ep}|$ represent the peak energies stored in the external magnetic and electric fields of the principal wave, respectively.

Upon integrating for the total complex power input and solving for the driving point impedance, equation (1) is replaced by the following equation

$$Z_{in} = lZ_i f_m{}^2 + j[X_p(b,c)-X_p(d,e)|f(d,e)|^2]$$

$$+ (30/jk)\int_e^d\int_e^d Re[f(s_1)^* f(s_2)]\left[\frac{\partial^2}{\partial s_1\partial s_2} - k^2\theta(s_1,s_2)\right]e(r_{12})ds_1 ds_2 \quad (9)$$

BICONICAL ANTENNA

Using the equation of continuity between the retarded scalar potential ϕ and the retarded vector potential $\bar{A}$, that is,

$$-j\omega\epsilon_0\phi = \nabla\cdot\bar{A} \quad (10)$$

if a real current distribution is postulated along an antenna of length $2l$, equation (9) may be written in the form

$$Z_{in} = 2lf_m{}^2 Z_i + j[X_p(0)-X_p(l)f(l)^2]$$

$$+ (j\omega\mu_0/4\pi)\int_{-l}^{l}\int_{-l}^{l} f(s_1)f(s_2)e(r_{12})d\bar{s}_1\cdot d\bar{s}_2 + 1/I_0\int_{-l}^{l} f(s_1)d\phi(s_1) \quad (11)$$

Integrating the last term by parts,

$$1/I_0^2 \left(\int_{-l}^{-0} + \int_{+0}^{l}\right) f(s_1)\,d\phi(s_1) = [\phi(l)f(l) - \phi(-l)f(-l)]/I_0 - [\phi(+0)-\phi(-0)]/I_0$$
$$- (1/I_0)\int_{-l}^{l} \phi(s_1) f(s_1)\,ds_1 \qquad (12)$$

Since the antenna is symmetrically fed, the current distribution is an even function and its derivative is an odd function. Also, in the case of a bi-conical antenna, Schelkunoff[3] has shown that the voltage wave depends only upon the principal wave while the current wave depends both upon the principal wave and the higher order waves. However, he also shows that the higher order currents vanish at the origin, and in fact, that they vanish identically for a vanishingly thin cone. Hence, since the total current vanishes at the ends of the antenna, in the limiting case,

$$[\phi(+0)-\phi(-0)]/I_0 = jX_p(0) \qquad (13)$$

and

$$[\phi(+l)-\phi(-l)]f(l)/I_0 = 0 \qquad (14)$$

Hence, again using the equation of continuity of potential, equation (12) becomes

$$1/I_0^2 \int_{-l}^{l} f(s_1)\,d\phi(s_1) = -jX_p(0) + 1/(j\omega\epsilon_0 I_0)\int_{-l}^{l} f'(s_1)\nabla_1\cdot\bar{A}\,ds_1 \qquad (15)$$
$$= -jX_p(0) - 1/(j\omega\epsilon_0 4\pi)\int_{-l}^{l}\int_{-l}^{l} f'(s_1)\, f(s_2)\,\frac{\partial e(r_{12})}{\partial s_1}\,ds_2\,ds_1$$

Integrating by parts,

$$1/I_0^2 \int_{-l}^{l} f(s_1)\,d\phi(s_1) = -jX_p(0) + 1/(j\omega\epsilon_0 4\pi)\int_{-l}^{l}\int_{-l}^{l} f'(s_1) f'(s_2) e(r_{12})\,ds_1 ds_2$$
$$-1/(j\omega\epsilon_0 4\pi)\int_{-l}^{l} f'(s_1)\left\{\left[f(s_2)\frac{\partial e(r_{12})}{\partial s_2}\right]_0^l + \left[f(s_2)\frac{\partial e(r_{12})}{\partial s_2}\right]_{-l}^0\right\} ds_1 \qquad (16)$$

Since the last integrand is an odd function, the single integral term on the right vanishes. Substituting from equation (16) into equation (11), it is seen that the reactive term $jX_p(0)$ is cancelled. Equation (11) may then be written

$$Z_{in} = 2lZ_i f_m^2 + 30/jk\int_{-l}^{l}\int_{-l}^{l} [f(s_1)f(s_2)]\,e(r_{12})\left[\frac{\partial^2}{\partial s_2\,\partial s_1} - k^2\cos\theta(s_1,s_2)\right] ds_1 ds_2 \quad (17)$$

Thus, assuming a perfect conductor and a sinusoidal current in the limiting form, equation (17) gives an expression from which Schelkunoff's inverse terminal impedance may be determined[3,4]. In other words, the asym-

---

4. J. G. Chaney, Simplification for mutual impedance of certain antennas', U. S. Naval Postgrad. School Tech. Rpt. no. 6, Nov., 1952.

totic form of the radiation impedance of a bi-conical antenna may be determined directly from the limiting case of the mutual impedance of a path along the axis and one along the perimeter without including the correction terms to the self impedance formula.

TERMINATED PARALLEL WIRE LINE

In the preceding case of the bi-conical antenna, the current distribution is such that it does not seem logical to assume the radiation impedance to be uniformly distributed along the antenna. However, for a low loss parallel wire transmission line terminated in its characteristic impedance , the current to a first approximation is uniformly distributed along the line. Hence, it does seem logical to distribute the radiation impedance uniformly along the transmission line, thus assuming that radiation losses produce an effect upon the current similar to that produced by losses due to the internal impedance of the wires.

Another distinction between an ordinary transmission line and the bi-conical antenna is that the generator must supply the complex power to the terminating impedance of the line as well as the complex power into the external fields characterized by the radiation impedance whereas no such terminating impedance exists for the bi-conical antenna considered as a transmission line. But this equivalent impedance accounts for the radiation losses, and hence it would not be proper also to introduce an attenuation into the current along the equivalent line for the bi-conical antenna as suggested above for the ordinary transmission line.

In determining the driving point impedance of a radiating transmission line terminated in its characteristic impedance Z_0, it has been shown[1] that the term $Z_0|f(d,e)|^2$ must be introduced into equation (9) in order that the lumped terminating impedance may be taken into consideration. Thus, for a transmission line of length l which has an internal impedance per unit loop length of Z_i, upon assuming that the radiation impedance Z_r has been computed from the double integral term of equation (9), the driving point impedance of the line may be expressed as

$$Z_{in} = l(Z_i+Z_r/l)f_m^2 + j[X_p(0)-X_p(l)|f(l)|^2] + Z_0|f(l)|^2 \qquad (18)$$

in which the shunt conductance has been assumed to be negligible.

For reasonably low losses, the attenuation constant α is given in transmission line theory as

$$\alpha = (R_r + lR_i)/(2lR_0) \tag{19}$$

For the customarily assumed current distribution given by

$$I(x) = I_0 e^{-(\alpha+j\beta)x} \tag{20}$$

with x being the distance from the generator, it follows that

$$|f(l)|^2 = e^{-2\alpha l}$$

$$f_m^2 = (1-e^{-2\alpha l})/2\alpha l \tag{21}$$

From equations (18), (19), and (21),

$$Z_{in} = R_0(lR_i+R_r)(1-e^{-2\alpha l})(R_r+lR_i) + Z_0 e^{-2\alpha l}$$
$$+jf_m^2(lX_i+X_r) + jX_0(1-e^{-2\alpha l}) \tag{22a}$$

or

$$Z_{in} = Z_0 + jf_m^2(lX_i+X_r) \tag{22b}$$

in which

$$Z_0 = [(lR_i+R_r+j\omega lL)(j\omega lC)]^{\frac{1}{2}} \tag{23}$$

and in which R_i, L, and C are the customary line parameters. Equation (22) shows that the radiation resistance affects the driving point impedance only through the modification of the characteristic impedance, whereas the reactive term associated with the radiation resistance appears to be distributed on a per unit loop length basis along with the internal reactance of the conductors. While the internal reactance per unit length is usually negligible, this is not necessarily true of the so called radiation reactance, that is, of the reactance due to the effects of retardation.

The radiation impedance of an open wire line terminated in its characteristic impedance and spaced ρ center to center has been given[5] as

---

5. J. G. Chaney, 'On the generalized circuit theory as applied to antennas and radiating lines'. U. S. Naval Postgrad. School Res. Paper no. 1, March, 1951.

$$Z_r/120 = Cin2kl - 2Cik\rho + Cik(r_0+l) + Cik(r_0-l) + \sin k\rho/k\rho - \cos kl \sin kr_0/kr_0 - (1-\sin 2kl/2kl) + j[Si2kl + 2Sik\rho - Sik(r_0+l) - Sik(r_0-l) - \cos kl\cos kr_0/kr_0 + (1+\cos 2kl)/2kl] \quad (24)$$

in which

$$r_0 \doteq (\rho^2+l^2)^{\frac{1}{2}}$$
$$Six = \int_0^x \sin t/t\,dt$$
$$Cix = -\int_x^\infty \cos t/t\,dt$$
$$Cinx = \int_0^x (1-\cos t)/t\,dt$$

For a long line with $l>>\rho$, equation (24) reduces to

$$Z_r/120 = (\sin k\rho/k\rho - 1) + 2Cink\rho + j2Sik\rho \quad (25)$$

If $k\rho << 1$,

$$Z_r = 60(k\rho)^2 + j240k\rho \quad (26)$$

and hence the radiation resistance of the line becomes

$$R_r = 240\pi^2(\rho/\lambda)^2 \quad (27)$$

Equation (27) checks with a value given by Storer and King[6] for a correspondingly special case.

Equation (18) may also be used for a line terminated in other than its characteristic impedance provided the standing wave ratio is not sufficiently large to invalidate the engineering approximation of uniformly distributing the radiation impedance. For this case, letting the coefficient of reflection be

$$K = |K|e^{j\psi} = \frac{Z_l - Z_0}{Z_l + Z_0} \quad (28)$$

the current square parameters become

$$|f(l)|^2 = e^{-2al}\,\frac{1+|K|^2-2|K|\cos\psi}{1+|K|^2e^{-4al}-2|K|e^{-2al}\cos(2\beta l-\psi)} \quad (29)$$

$$f_m^2 = \frac{(1/2al)(1-e^{-2al})(1+|K|^2e^{-2al})-2|K|e^{-2al}\cos(\beta l-\psi)\sin\beta l/\beta l}{1+|K|^2e^{-4al}-2|K|e^{-2al}\cos(2\beta l-\psi)} \quad (30)$$

---

6. R. E. Storer and R. W. P. King, 'Radiation resistance of a two wire line', Cruft Lab. Tech. Rpt. no. 69, March, 1949

respectively.

CONCLUSION

It thus appears that whereas the correction terms introduced into equation (9) may be cancelled by a proper integration procedure when finding the asymtotic radiation impedance of a biconical antenna, they must be retained for the problem of determining the driving point impedance of a radiating transmission line. If the standing wave ratio for a transmission line is not too large, the procedure indicated by equations (18) and (19) may still be followed for finding the effects on the driving point impedance caused by the radiation impedance. The standing wave ratio for the principal current of a 600 ohm bi-conical antenna is about nine to one. Hence, the procedure is not valid for such an antenna.

It should be pointed out that equation (22) may be used for determining the driving point impedance of a rhombic antenna provided due cognizance is taken of the non-uniformity of the line.

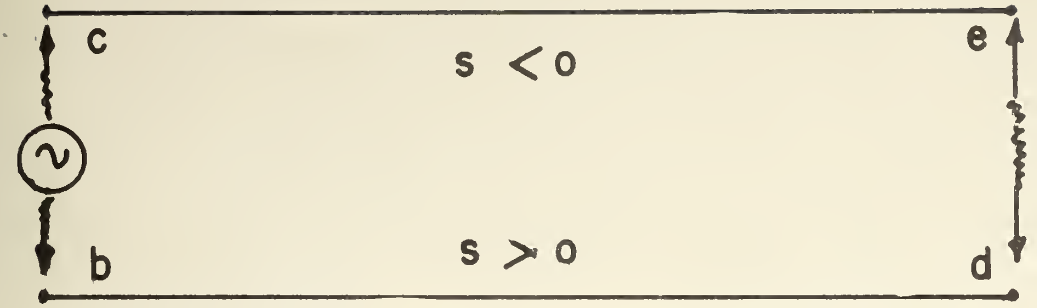

Figure 1. A generalized circuit

27944

Chaney
On the impedance of a
radiating circuit.

TA7
.U64
no.9

27944

Chaney
On the impedance of a radiating
circuit.